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Simple transformation functions for finding better minima[☆]

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Abstract

This work presents two transformation functions, the α-function and the M-function, for finding better minimizers in global optimization. We prove that under some general assumptions these functions possess the characters of both tunnelling functions and filled functions. Numerical tests from some test functions show that our transformation functions are very effective in finding better minima.

Keywords: Mathematical programming; Global optimization; Filled function; Tunnelling function

1. Introduction

Because of advances in science, economics and engineering, studies on global optimization for the multi-minimum nonlinear programming problem

$$\min\{f(x) : x \in R^n\}$$

have become a topic of great concern. Since the 1970's, many new theoretical and computational contributions such as Dixon and Szegö [1]; Horst, Pardalos and Thoai [6] and so on have been developed. In particular, the tunnelling function proposed by Levy and Montalvo [7] and the filled function introduced by Ge and Qin [2] are two practical and useful tools for global optimization.

The tunnelling algorithm is composed of two phases, a minimizing phase and a tunnelling phase. These two phases are used alternately to search for the better minimizer. Suppose minimizers $x_1^*, \ldots, x_{m-1}^*$ have been found. In the first phase, a classical algorithm such as Newton's method or the steepest descent method can be used to find next local minimizer x_m^* of the objective function $f(x)$. In the second phase, the search for roots of a defined auxiliary function, the tunnelling function $T(x)$, is carried out. The tunnelling function in Levy and Montalvo [7] and Yao [10] is defined

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as

$$T(x) = \frac{f(x) - f(x_m^*)}{[(x - x_1^*)^T (x - x_1^*)]^{\alpha_1} \cdots [(x - x_m^*)^T (x - x_m^*)]^{\alpha_m}}. \tag{1}$$

If we can get $\bar{x}$ such that $\bar{x} \notin \{x_1^*, \ldots, x_m^*\}$ and $T(\bar{x}) \leq 0$, i.e., $f(\bar{x}) \leq f(x_m^*)$, then $\bar{x}$ is a new starting point for the next iteration. The denominator of (1) is a pole at x_i^* $(i = 1, \ldots, m)$ with power α_i which prevents x_i^* from being a zero of the tunnelling function.

The filled function method is essentially similar to that for the tunnelling function. The only difference is the filled function that is used for finding $\bar{x}$ in phase II. There are various filled functions according to their expressions. The following filled function was applied by Han and Han [5]:

$$P(x, r, \rho) = \frac{1}{r + f(x)} \exp\left(-\frac{\|x - x^*\|^2}{\rho}\right), \tag{2}$$

and another filled function with prefixed point x_0

$$U(x, A, h) = \eta(\|x - x_0\|)\varphi(A[f(x) - f(x^*) + h]) \tag{3}$$

was proposed by Ge and Qin [4], Zhang, Ng, Li and Tian [11] and further discussed by Lucidi and Piccialli [9].

No matter what the expressions for the filled functions are (such as (2) and (3)), they possess the following common properties:

(i) x^* is a maximizer of the filled functions,
(ii) the filled function has no stationary point in $\{x : f(x) \geq f(x^*), x \neq x^*\}$, and
(iii) if $f(x^*)$ is not a global minimum, the filled function has either a minimizer in $\{x : f(x) < f(x^*)\}$ or a stationary point along the ray $x^* \to x'$, where x' is in a lower basin.

In general, there are two difficulties in global optimization. The first is how to leave the local minimizer to go to a better one. The second is how to check whether the current minimizer is a global solution of the problem. We pay our main attention to the first issue in this work.

This work is organized as follows. In Section 2, we introduce our problem and some assumptions. In Sections 3 and 4, we present the α-function and the M-function, and discuss their properties. In Section 5, we state our algorithm based on the α-function or M-function and make a numerical test. Last, in Section 6, we give our conclusion.

2. Problem and assumptions

Consider the unconstrained optimal problem

$$\min\{f(x): x \in \Omega\} \tag{4}$$

where $f : R^n \to R$, and $\Omega \subset R^n$ is a large enough bounded closed region. We need the following assumptions:

Assumption 1. $f(x)$ is continuously differentiable in R^n and there exists a $K > 0$ such that $\|\nabla f(x)\| \leq K, \forall x \in R^n$.

Assumption 2. $f(x) \to +\infty$ as $\|x\| \to +\infty$, namely, $f(x)$ is a coercive function.

Assumption 3. $f(x)$ has a finite number of different minimal function values.

Assumption 3 is weaker compared with the assumptions used in many papers, such as [3,7] which assumes that $f(x)$ has a finite number of minimizers. Assumption 2 implies that there exists a bounded closed domain $\Omega \subset R^n$ that contains all minimizers of $f(x)$ in R^n. We only need to consider a bounded closed domain Ω because all minimizers of problem (4) are the inner points of Ω. So, the problem (4) is equivalent to

$$\min\{f(x): x \in R^n\}.$$

Suppose that x_1^* is a local minimizer but not a global one. If there is an auxiliary function based on x_1^* that can be used to force the sequence of iterative points to leave the basin containing x_1^* to go to another point x' which is contained in a lower basin such that $f(x') \leq f(x_1^*)$, then we can obtain a better local minimizer x_2^* from x'. This auxiliary function is called the *transformation function*. In the work, we present two transformation functions which are simpler and possess the main characteristics of a filled function.

Let $L(P)$ and $G(P)$ be the set of local minimizers and global minimizers of problem (4) respectively. Let $x^* \in L(P)$ and $x_G^* \in G(P)$. Moreover, let r satisfy

$$0 < r < \min\{|f(x_1^*) - f(x_2^*)| : f(x_1^*) \neq f(x_2^*), x_1^*, x_2^* \in L(P)\}. \tag{5}$$

Then, r is a parameter depending on the problem. We split the region $\Omega \setminus \{x^*\}$ into two subsets:

$$S_1 = \{x : f(x) \geq f(x^*), x \in \Omega \setminus \{x^*\}\},$$
$$S_2 = \{x : f(x) < f(x^*), x \in \Omega\}.$$

Hence, $\Omega = S_1 \bigcup S_2 \bigcup \{x^*\}$. Actually, S_1 and S_2 are the functions of x^*. Lastly, let

$$\bar{D} = \max_{x \in \Omega} \|x - x^*\|. \tag{6}$$

3. The α-function and its properties

The *α-function* is defined as

$$Q(x, x^*, r, \alpha) = \frac{f(x) - f(x^*) + r}{[r + \|x - x^*\|]^\alpha}, \tag{7}$$

where $\alpha \geq 1$, r is from (5) and $x^* \in L(P)$.

It is easy to see that $Q(x^*, x^*, r, \alpha) = \frac{1}{r^{\alpha-1}} \neq 0$. If there exists x such that $Q(x, x^*, r, \alpha) = 0$, there must be $f(x) < f(x^*)$ owing to $r > 0$. Therefore, function $Q(x, x^*, r, \alpha)$ possesses the feature of a tunnelling function. We call it a *transformative tunnelling function*. The gradient of $Q(x, x^*, r, \alpha)$ is

$$\nabla Q(x, x^*, r, \alpha) = \frac{1}{[r + \|x - x^*\|]^\alpha} \left\{ \nabla f(x) - \frac{\alpha[f(x) - f(x^*) + r](x - x^*)}{(r + \|x - x^*\|)\|x - x^*\|} \right\}. \tag{8}$$

Obviously, $\nabla Q(x, x^*, r, \alpha)$ is continuous in $\Omega \setminus \{x^*\}$ from the continuous differentiability of $f(x)$. $Q(x, x^*, r, \alpha)$ is subdifferentiable at x^*. Let

$$e_x = \frac{x - x^*}{\|x - x^*\|}. \tag{9}$$

We will show that $Q(x, x^*, r, \alpha)$ has some characteristics of the filled functions.

Theorem 1. *Suppose that Assumption 1 hold and $\alpha > \frac{K(r+\bar{D})}{r}$. Then, for all $x \in S_1$,*

$$(x - x^*)^T \nabla Q(x, x^*, r, \alpha) < 0, \tag{10}$$

where x_1^* is a local minimizer but not a global one.

Proof. That Assumption 1 holds implies (8) is true. For any $x \in S_1$, i.e. $f(x) \geq f(x^*)$, we have from (9) that

$$\begin{aligned} e_x^T \nabla Q(x, x^*, r, \alpha) &= \frac{1}{[r + \|x - x^*\|]^\alpha} \left[e_x^T \nabla f(x) - \frac{\alpha(f(x) - f(x^*) + r)}{r + \|x - x^*\|} \right] \\ &\leq \frac{1}{[r + \|x - x^*\|]^\alpha} \left(\|\nabla f(x)\| - \frac{\alpha r}{r + \|x - x^*\|} \right) \\ &\leq \frac{1}{[r + \|x - x^*\|]^\alpha} \left(K - \frac{\alpha r}{r + \bar{D}} \right) < 0, \end{aligned}$$

which implies that expression (10) holds. We finish the proof. ∎

Theorem 1 illuminates that for any $x \in S_1$, namely $f(x) \geq f(x^*)$, $x - x^*$ is a strictly descending direction of function $Q(x, x^*, r, \alpha)$ at point x when α is suitably large. So, the following corollary is evident.

Corollary 1. *Under the conditions of Theorem 1, the α-function $Q(x, x^*, r, \alpha)$ has no stationary point in S_1.*

The theorem below indicates that the farther the point x gets from x^*, the less the value of $Q(x, x^*, r, \alpha)$ is in S_1.

Theorem 2. *Suppose that $x_1, x_2 \in S_1$ and there is an $\epsilon > 0$ such that $\|x_1 - x^*\| \geq \|x_2 - x^*\| + \epsilon$. If*

$$\alpha > \max\left\{1, \frac{\ln r - \ln(K\bar{D} + r)}{\ln(\bar{D} + r - \epsilon) - \ln(\bar{D} + r)}\right\}, \tag{11}$$

then $Q(x_1, x^*, r, \alpha) < Q(x_2, x^*, r, \alpha)$.

Proof. Let $x_1, x_2 \in S_1$, i.e. $f(x_1) \geq f(x^*)$ and $f(x_2) \geq f(x^*)$. We will discuss the following two cases: (i) $f(x^*) \leq f(x_1) \leq f(x_2)$, and (ii) $f(x^*) \leq f(x_2) \leq f(x_1)$. In case (i), since $\alpha > 1$, we have

$$\begin{aligned} Q(x_1, x^*, r, \alpha) &= \frac{f(x_1) - f(x^*) + r}{[r + \|x_1 - x^*\|]^\alpha} \\ &\leq \frac{f(x_2) - f(x^*) + r}{[r + \|x_1 - x^*\|]^\alpha} \\ &< \frac{f(x_2) - f(x^*) + r}{[r + \|x_2 - x^*\|]^\alpha} \\ &= Q(x_2, x^*, r, \alpha). \end{aligned}$$

In case (ii), since $\|x_1 - x^*\| \geq \|x_2 - x^*\| + \epsilon$, we have

$$\frac{r + \|x_2 - x^*\|}{r + \|x_1 - x^*\|} \leq 1 - \frac{\epsilon}{\bar{D} + r}.$$

From the assumptions $\|\nabla f(x)\| \leq K$ and $f(x_2) \geq f(x^*)$, we can obtain that for $0 < \theta < 1$,

$$\begin{aligned} \frac{f(x_2) - f(x^*) + r}{f(x_1) - f(x^*) + r} &\geq \frac{r}{\nabla f[x^* + \theta(x_1 - x^*)]^T (x_1 - x^*) + r} \\ &\geq \frac{r}{K\bar{D} + r}. \end{aligned}$$

Considering $\alpha > \frac{\ln r - \ln(K\bar{D}+r)}{\ln(\bar{D}+r-\epsilon) - \ln(\bar{D}+r)}$, we have

$$\frac{r}{K\bar{D} + r} > \left(1 - \frac{\epsilon}{\bar{D} + r}\right)^\alpha.$$

Thus, we obtain

$$\frac{f(x_2) - f(x^*) + r}{f(x_1) - f(x^*) + r} > \left(\frac{r + \|x_2 - x^*\|}{r + \|x_1 - x^*\|}\right)^\alpha.$$

This implies $Q(x_1, x^*, r, \alpha) < Q(x_2, x^*, r, \alpha)$. We finish the proof. ■

Corollary 2. *Let $x \in S_1$ and let there exist an $\epsilon > 0$ such that $\|x - x^*\| \geq \epsilon$. Then, $Q(x, x^*, r, \alpha) < Q(x^*, x^*, r, \alpha)$ if (11) is true.*

Proof. Let $x_1 = x$, $x_2 = x^*$ and $f(x^*) = f(x_2) \leq f(x_1)$. Then all inequalities in the proof for the second case of Theorem 2 hold. Therefore, the conclusion in the corollary is correct. We finish the proof. ■

Now we prove that the local minimizer of $f(x)$ is a global maximizer of the transformative function $Q(x, x^*, r, \alpha)$.

Theorem 3. *x^* is a global maximizer of the transformative function $Q(x, x^*, r, \alpha)$ for suitably large α.*

Proof. Obviously, $Q(x^*, x^*, r, \alpha) = \frac{1}{r^{\alpha-1}}$. We only needed to prove that

$$Q(x, x^*, r, \alpha) < Q(x^*, x^*, r, \alpha) \tag{12}$$

holds for any $x \in \Omega \setminus \{x^*\}$. We discuss the cases of $x \in S_1$ and $x \in S_2$.

In the case of $x \in S_1$, from Taylor expansion,

$$Q(x^*, x^*, r, \alpha) - Q(x, x^*, r, \alpha) = -(x - x^*)^T \nabla Q(x, x^*, r, \alpha) + o(\|x^* - x\|).$$

Let $N(x^*)$ be a neighborhood of x^*. When $x \in N(x^*)$ but $x \neq x^*$, i.e. $\|x - x^*\|$ is small, and $\alpha > \frac{K(r+\bar{D})}{r}$, the inequality (12) follows from Theorem 1. When $x \in S_1 \setminus N(x^*)$, there must exist an $\epsilon > 0$ such that $\|x - x^*\| > \epsilon$. So, the inequality (12) holds from Corollary 2 since α satisfies inequality (11).

In the case of $x \in S_2$, i.e., $f(x) < f(x^*)$ and $\|x - x^*\| > 0$, we also have

$$\begin{aligned} Q(x, x^*, r, \alpha) &= \frac{f(x) - f(x^*) + r}{[r + \|x - x^*\|]^\alpha} \\ &< \frac{r}{[r + \|x - x^*\|]^\alpha} \\ &< \frac{r}{r^\alpha} \\ &= Q(x^*, x^*, r, \alpha). \end{aligned}$$

We finish the proof. ■

Theorem 4. *If $x^* \in L(P)$ but $x^* \notin G(P)$, then there exists a minimizer of $Q(x, x^*, r, \alpha)$ in the region S_2.*

Proof. Note that $S_2 \neq \emptyset$ since x^* is not a global minimizer of (4), and $Q(x_G^*, x^*, r, \alpha) < 0$ from the definitions of $Q(x, x^*, r, \alpha)$ and r. Because $Q(x, x^*, r, \alpha)$ is continuous in the bounded closed domain Ω, there exists $\bar{x}$ which is a global minimizer of $Q(x, x^*, r, \alpha)$ such that

$$Q(\bar{x}, x^*, r, \alpha) \leq Q(x_G^*, x^*, r, \alpha) < 0.$$

This implies that $f(\bar{x}) < f(x^*)$ from (7), namely $\bar{x} \in S_2$. We finish the proof. ■

Moreover, we have the following two theorems.

Theorem 5. *If $x^*, x_1^* \in L(P)$ and $f(x_1^*) < f(x^*)$, then there is a neighborhood of $x_1^* N(x_1^*, \epsilon_1)$ ($\epsilon_1 > 0$) such that $Q(x, x^*, r, \alpha) \leq 0$, $\forall x \in N(x_1^*, \epsilon_1)$.*

Proof. We omit the proof since it is similar to that for Theorem 4. ■

Theorem 6. *If x^* is a local minimizer of (4) but not a global one, then for suitably large α, there is an $\bar{x} \in S_2$ such that*

$$(\bar{x} - x^*)^T \nabla Q(\bar{x}, x^*, r, \alpha) = 0.$$

Proof. Because x^* is not a global minimizer, S_2 is not empty. Let

$$\phi(x) = e_x^T \nabla f(x) - \frac{\alpha(f(x) - f(x^*) + r)}{r + \|x - x^*\|}. \tag{13}$$

It is easy to see that

$$\begin{aligned} (x - x^*)^T \nabla Q(x, x^*, r, \alpha) &= \frac{\|x - x^*\|}{[r + \|x - x^*\|]^\alpha} \left[e_x^T \nabla f(x) - \frac{\alpha(f(x) - f(x^*) + r)}{r + \|x - x^*\|} \right] \\ &= 0 \end{aligned}$$

if and only if $\phi(x) = 0$.

Let x_G^* be a global minimizer of problem (4). Then, we learn that $\phi(x_G^*) > 0$ from (5) and $\nabla f(x_G^*) = 0$. On the other hand, viewing the proof of Theorem 1, we have $\phi(x) < 0$ when $x \in S_1$ and $\alpha > \frac{K(r+\bar{D})}{r}$. Thus, from the continuity of $\phi(x)$, there exists $\bar{x} \neq x^*$ such that $\phi(\bar{x}) = 0$. Since $\bar{x} \notin S_1$ from Theorem 1, we obtain $\bar{x} \in S_2$. We finish the proof. ■

4. The M-function and its properties

Let $M \geq K$ be a constant. The *M-function* is defined as

$$P(x, x^*, r) = \frac{f(x) - f(x^*) + r}{r + M\|x - x^*\|}. \tag{14}$$

From (14), we can observe that if there is $\bar{x} \neq x^*$ such that $P(\bar{x}, x^*, r) = 0$, then $f(\bar{x}) < f(x^*)$. Therefore, $P(x, x^*, r)$ is also a transformative tunnelling function. The gradient of $P(x, x^*, r)$ is

$$\nabla P(x, x^*, r) = \frac{1}{r + M\|x - x^*\|}\left[\nabla f(x) - \frac{M(f(x) - f(x^*) + r)}{(r + M\|x - x^*\|)} \cdot \frac{x - x^*}{\|x - x^*\|}\right]. \tag{15}$$

$\nabla P(x, x^*, r)$ is obviously continuous in $\Omega \setminus \{x^*\}$.

Theorem 7 will show that x^* is a global maximizer of $P(x, x^*, r)$. Thus, $P(x, x^*, r)$ is concave in a neighborhood of x^* and its subdifferential exists at x^*.

Theorem 7. *If Assumption 1 holds and $M \geq K$, then $P(x, x^*, r) \leq P(x^*, x^*, r)$ for any $x \neq x^*$.*

Proof. It is obvious that $P(x^*, x^*, r) = 1$. On the basis of Assumption 1 and the differential mean value theorem, we obtain

$$|P(x, x^*, r)| = \frac{|f(x) - f(x^*) + r|}{r + M\|x - x^*\|} \leq \frac{|f(x) - f(x^*)| + r}{r + M\|x - x^*\|} \leq \frac{M\|x - x^*\| + r}{r + M\|x - x^*\|} = 1.$$

This implies that $P(x, x^*, r) \leq |P(x, x^*, r)| \leq P(x^*, x^*, r)$ for $x \neq x^*$. We finish the proof. ■

Theorem 8. (a) *Let x^* be a local minimizer of problem (4). If $\nabla f(x) \neq MP(x, x^*, r)e_x$ for any $x \in S_1$, then $\nabla P(x, x^*, r) \neq 0$.*

(b) *If x^* is a local minimizer of (4) but not a global one, there exists a minimizer of $P(x, x^*, r)$ in S_2.*

Proof. The conclusion of part (a) is clear from (15). We prove part (b) now. Since x^* is not a global minimizer of $f(x)$, S_2 is not empty. Recalling the definition of $P(x, x^*, r)$ and formula (5), we have $P(x^*_G, x^*, r) < 0$. Due to Assumption 1 and (15), $P(x, x^*, r)$ is continuously differentiable in $\Omega \backslash \{x^*\}$. Furthermore, from Theorem 7, x^* is a maximizer of $P(x, x^*, r)$. Therefore, there exists a minimizer $\bar{x}$ in the closed domain $\Omega \backslash N(x^*)$ where $N(x^*)$ is a neighborhood of x^* such that

$$P(\bar{x}, x^*, r) \leq P(x^*_G, x^*, r) < 0.$$

This implies that $f(\bar{x}) - f(x^*) + r < 0$ or, furthermore, $f(\bar{x}) < f(x^*)$. Namely, $\bar{x} \in S_2$. We finish the proof. ■

If $\bar{x}$ is an inner point of Ω, then we have following corollary.

Corollary 3. *Under the conditions of Theorem 8*(b), *there exists $x \in S_2$ such that $\nabla P(x, x^*, r) = 0$.*

5. Algorithm and numerical tests

In the section, we state the algorithm for global optimization and our numerical tests based on the algorithm.

Algorithm:

1. Let $r, \beta, \delta \in (0, 1)$, $\alpha \geq 1$, $t > 1$, and $\varepsilon > 0$. Select initial point $x^0_1 \in \Omega$ and set $k = 1$.
2. Compute $x^*_k = \arg\min f(x)$ based on initial point x^0_k.
3. Choose $2n$ random directions $e^k_1, \ldots, e^k_{2n}$ and set $i = 1$.
4. Compute $x_i = \arg\min_x Q(x, x^*_k, r, \alpha)$ by taking $x^*_k + \delta e_i$ as the initial point. If $Q(x_i, x^*_k, r, \alpha) \leq 0$, let $x^0_{k+1} = x_i$, $k = k + 1$ and go to Step 2.

Table 1
Function (i)

k	x_k^0	x_k^*	$f(x_k^*)$
1	(−2,−1)	(−1.9283030, −0.8061283)	0.5106881
2	(0.4842082, 0.2864985)	(0.1956660, 0.7190378)	−0.9894518
3	(−1.1314399E−02, −0.6862821)	(−8.9833766e−02, −0.7126262)	−1.031628

Table 2
Function (ii)

k	x_k^0	x_k^*	$f(x_k^*)$
1	(1, 1)	(0.0000471, −0.9999888)	3.0000005

5. If $i = 2n$ and $r < \varepsilon$, then stop. x_k^* is the global optimization solution. If $i < 2n$, then let $i = i + 1$ and go to Step 4. If $i = 2n$ and $r \geq \varepsilon$, then let $r = \beta r$, $\alpha = t\alpha$ and go to Step 3.

Now we make some numerical tests to show the tunnelling function can be used as a filled function in phase II. The transformation function used in our tests is (7),

$$Q(x, x^*, r, \alpha) = \frac{f(x) - f(x^*) + r}{[r + \|x - x^*\|]^\alpha}.$$

Parameter $r > 0$ can be regarded as a user-defined tolerance by adjusting the value of r in the computing process. If the root of $Q(x, x^*, r, \alpha)$ cannot be found while $r > 0$ is small enough, then the current local minimizer is considered as a global minimizer. On the basis of this point, parameter r plays a key role in overcoming the second difficulty mentioned in the first section.

We choose the following five functions for our test.

(i) *Six-hump camel back function* [4]:

$$f(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 - x_1x_2 - 4x_2^2 + 4x_2^4, \quad -3 \leq x_1, x_2 \leq 3.$$

The global minimum solutions: $x_G^* = (0.0898, 0.7127)$ or $(-0.0898, -0.7127)$ and $f_G^* = -1.0316$.

(ii) *Goldstein and Price function* [4]:

$$f(x) = [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)]$$
$$\times [30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)], \quad -3 \leq x_1, x_2 \leq 3.$$

It has unique global minimizer $x_G^* = (0, -1)$, and $f(x_G^*) = 3$.

(iii) *Treccani function* [4]:

$$f(x) = x_1^4 + 4x_1^3 + 4x_1^2 + x_2^2, \quad -3 \leq x_1, x_2 \leq 3.$$

The global minimum solutions: $x_G^* = (0.0000, 0.0000)$ or $(-2.0000, 0.0000)$ and $f_G^* = 0.0000$.

(iv) *Rastrigin* [8]:

$$f(x) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2), \quad -1 \leq x_1, x_2 \leq 1.$$

The global minimum solution: $x_G^* = (0.0000, 0.0000)$ and $f_G^* = -2.0000$.

(v) *Two-dimensional function* [6]:

$$f(x) = [1 - 2x_2 + c\sin(4\pi x_2) - x_1]^2 + [x_2 - 0.5\sin(2\pi x_1)]^2, \quad -10 \leq x_1, x_2 \leq 10,$$

where $c = 0.2, 0.5$. $f(x)$ has the unique global minimal value $f(x_G^*) = 0$ for all c, but it has many global minimizers.

We list our computational results in Tables 1–5.

Table 3
Function (iii)

k	x_k^0	x_k^*	$f(x_k^*)$
1	(−2, 1)	(−2.000149, 4.9991675E−02)	2.4992561E−03
2	(−1.994570, 1.3198899E−02)	(−1.994574, 1.3196260E−02)	2.9125065E−04
3	(−7.2052111E−03,−9.0170018E−03)	(−7.1995091E−03,−9.0151988E−03)	2.8711554E−04

Table 4
Function (iv)

k	x_k^0	x_k^*	$f(x_k^*)$
1	(0.8,0.8)	(0.6938895, 0.6938895)	−1.031207
2	(0.4040276, −0.3131202)	(0.3469777, −0.3468937)	−1.757801
3	(−0.3239264, −1.3505705E−02)	(−0.3468803, −2.4708723E−05)	−1.878900
4	(−1.1737966E−02, 1.9938093E−02)	(−3.1793712E−05, 5.4679724E−05)	−1.999999

Table 5
Function (v) ($c = 0.2$)

k	x_k^0	x_k^*	$f(x_k^*)$
1	(0,0)	(4.1562073e−02, −9.4807312e−02)	0.5174555
2	(2.693728, −0.9261180)	(0.5527265, −0.1039423)	3.3222079E−02
3	(4.7956310E−02, 0.2966070)	(0.2300172, 0.5541206)	3.9381064E−03
4	(−0.9116482, 0.8058445)	(0.1025250, 0.3005036)	5.3706035E−08

6. Conclusion

In this work, we present two transformation functions, the α-function and the M-function. We prove under some general assumptions that these two functions have some properties of tunnelling functions and filled functions. Last, we choose some well-known test functions and make a numerical test on these functions. The numerical results show that our transformation functions are very effective in finding better minima.

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